Appl Math Optim (2017) 75:175–192
DOI 10.1007/s00245-015-9327-z

On Asymptotic Estimation of a Discrete Type C_0-Semigroups on Dense Sets: Application to Neutral Type Systems

G. M. Sklyar[1] · P. Polak[1]

Published online: 22 January 2016

Abstract We consider an abstract Cauchy problem for a certain class of linear differential equations in Hilbert space. We obtain a criterion for stability on some dense subsets of the state space of the C_0-semigroups in terms of location of eigenvalues of their infinitesimal generators (so-called polynomial stability). We apply this result to analysis of stability and stabilizability of special class of neutral type systems with distributed delay.

Keywords Polynomial stability · Asymptotic behavior of solutions · Delay systems · Neutral type systems

Mathematics Subject Classification 34K40 · 34K20 · 47D03 · 47D06

1 Introduction

An important problem in the theory of differential equations is to determine the asymptotic behavior of solutions. One of the main issues in this topic concern stability. In the case of finite dimensional linear systems (exponential or asymptotic) stability of a system is equivalent to the fact that all eigenvalues are in the open left half-plane. For linear equations in infinite-dimensional space the problem of stability is much more complicated. In particular, a system can be asymptotically stable even if it possesses a point of spectrum on the imaginary axis (see Arendt and Batty [1], Lyubich and

✉ G. M. Sklyar
sklar@univ.szczecin.pl

P. Polak
piotr.polak@wmf.univ.szczecin.pl

1 Institute of Mathematics, University of Szczecin, Wielkopolska 15, 70-451 Szczecin, Poland

Phong [8], Sklyar and Shirman [17]). On the other hand, the system may be unstable even in the case when it is spectrum is contained in the open left half-plane, while the eigenvalues approach imaginary axis. Such a situation may occur for hyperbolic equations or delay equations of neutral type (e.g. Rabah et al. [13,14]).

One of important characteristics describing the asymptotic behavior of solutions of a linear differential equation is the growth bound $\omega_0 = \omega_0(T) = \lim_{t\to+\infty} \frac{1}{t} \ln \|T(t)\|$ of a corresponding C_0-semigroup $T(t)$. In the context of stability the critical situation is when $\omega_0 = 0$. In this case the semigroup cannot be exponentially stable ($\|T(t)\| \not\to 0$), but it still may be asymptotically stable ($\|T(t)x\| \to 0,\ x \in X$). Then the solutions tend to zero arbitrarily slow. However, the solutions with sufficiently regular initial states (for example, from the domain of the generator) tend to zero uniformly, i.e. $\|T(t)A^{-1}\| \to 0$. In particular, Batty [3,4] and Phong [10] (cf. Sklyar [15]) proved that for bounded C_0-semigroup $T(t)$ on a Banach space with the generator A, the following holds: if

$$\sigma(A) \cap (i\mathbb{R}) = \emptyset,$$

then

$$\|T(t)A^{-1}\| \to 0, \quad t \to +\infty.$$

This means that all solutions of abstract Cauchy problem with initial condition in the domain of A tend uniformly to zero not slower then the function $\|T(t)A^{-1}\|$. This fact leads to the concept of polynomial stability:

Definition 1.1 [2] We say the semigroup $T(t)$ generated by A is polynomially stable if there exist constants $\alpha, \beta, C > 0$ such that

$$\|T(t)(A - dI)^{-\alpha}\| \le Ct^{-\beta}, \quad t > 0, \tag{1}$$

for some $d \in \rho(A)$.

It is easy to see that the above definition does not depend on the choice of d. Hence if $0 \in \rho(A)$ then (1) is equivalent to

$$\|T(t)A^{-\alpha}\| \le Ct^{-\beta}, \quad t > 0. \tag{2}$$

The rate of decay of solutions with an initial condition from the set $D(A^{-1})$ and in general in the set $D(A^{-\alpha}),\ \alpha > 0$, is closely related to the asymptotic behavior of the resolvent $R(A, \lambda)$ on the imaginary axis (Bátkai et al. [2]; Borichev and Tomilov [5]). In particular, it is shown in [5, Theorem 2.4] that for a bounded C_0-semigroup $T(t)$ on a Hilbert space H and some positive, fixed constant α, the following conditions are equivalent:

$$\begin{aligned} &\text{(i)} \quad \|R(A, is)\| = O(|s|^{\alpha}), \quad s \to \infty,\\ &\text{(ii)} \quad \|T(t)A^{-\alpha}\| = O(t^{-1}), \quad t \to +\infty,\\ &\text{(iii)} \quad \|T(t)A^{-1}\| = O(t^{-\frac{1}{\alpha}}), \quad t \to +\infty. \end{aligned}$$

It is shown in [2] that the rate of growth of the resolvent on imaginary axis implies some restrictions on the location of the spectrum. In particular, if condition (i) holds for the generator of a bounded C_0-semigroup acting in Banach space, then the spectrum of the generator satisfies the following condition (see [2, Propositions 3.6, 3.7]):

$$(A) \quad \operatorname{Re}\lambda \le -\gamma |\operatorname{Im}\lambda|^{-\alpha} : \lambda \in \sigma(\mathcal{A})$$

for some real, positive constant γ and small values of $|\operatorname{Re}\lambda|$. However, in general case the knowledge about location of the spectrum is not enough to determine asymptotic behavior of the resolvent and/or the semigroup. At the same time in [2, Proposition 4.1] it is also shown that if the generator A of a bounded semigroup is a normal operator in Hilbert space with spectrum $\sigma(A)$ in the open left half-plane, then the condition (ii) is equivalent to (A).

In this paper we conduct the analysis of polynomial stability in case of certain class of not necessarily bounded discrete semigroups acting in Hilbert space extending the results of [2,5] to this class. Namely we consider the the class of semigroups whose generator has spectrum splitted into a family of finite separated sets and corresponding eigenspaces are finite dimensional and form a Riesz basis. We give an estimation of asymptotic behavior of these semigroups on dense sets (like $D(A^{-\alpha})$) depending on asymptotic closeness of the eigenvalues to the vertical line $\operatorname{Re}\lambda = \omega_0$. In particular, in the case of bounded semigroups of our class we show that the condition (A) is equivalent to (iii). The class of semigroups mentioned above was considered earlier in the paper of Miloslavskii [9], where the estimation of the semigroup norm was obtained (see [9, Theorem 1]).

Such semigroups appear naturally in the analysis of delay equation of neutral type. Following [12,13] we consider equation

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + \int_{-1}^{0} A_2(\theta)\dot{z}(t+\theta)d\theta + \int_{-1}^{0} A_3(\theta)z(t+\theta)d\theta, \tag{3}$$

where A_{-1} is a $n \times n$ invertible complex matrix, A_2 and A_3 are $n \times n$ matrices of functions from $L_2(-1, 0)$. The Eq. (3) can be rewritten in the operator form

$$\dot{x} = \mathcal{A}x, \qquad x \in \mathcal{M}_2, \tag{4}$$

where $\mathcal{M}_2 = \mathbb{C}^n \times L_2(-1, 0; \mathbb{C}^n)$, the operator $\mathcal{A}$ is then given by

$$\mathcal{A}\begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \begin{pmatrix} \int_{-1}^{0} A_2(\theta)\dot{z}_t(\theta)d\theta + \int_{-1}^{0} A_3(\theta)z_t(\theta)d\theta \\ dz_t(\theta)/d\theta \end{pmatrix}, \tag{5}$$

where $z_t(\cdot) = z(t + \cdot)$ and the domain of $\mathcal{A}$ is as follows:

$$\mathcal{D}(\mathcal{A}) = \left\{(y, z(\cdot)) : z \in H^1(-1, 0; \mathbb{C}^n),\, y = z(0) - A_{-1}z(-1)\right\} \subset \mathcal{M}_2. \tag{6}$$

This model of neutral type equation was introduced by Burns et al. [6]. The complete spectral analysis of the operator (5)–(6) was given in [13]. In particular, it was shown that the operator $\mathcal{A}$ is a generator of discrete C_0-group, whose spectrum consists of eigenvalues only, that lie asymptotically close to some vertical lines and can be grouped into finite, separated families. Riesz projections corresponding to these families generate Riesz basis of subspaces in the space $\mathcal{M}_2$. We generalize these properties and consider the following abstract class of operators. Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be the generator of C_0-semigroup in Hilbert space $\mathcal{H}$. We assume that

(B1) $\sigma(\mathcal{A}) = \bigcup_{k\in\mathbb{Z}} \sigma_k$, and $\inf\{|\lambda - \mu| : \lambda \in \sigma_i, \mu \in \sigma_j, i \neq j\} = d > 0$;
(B2) $\dim \mathcal{P}_k\mathcal{H} \leq N, k \in \mathbb{Z}$, where $\mathcal{P}_k$ is a spectral projection corresponding to σ_k;
(B3) subspaces $V_k := \mathcal{P}_k\mathcal{H}, k \in \mathbb{Z}$, constitute Riesz basis of subspaces.

Note that the condition (B2) implies that the families σ_k must be finite and $\#\sigma_k \leq N$ for all $k \in \mathbb{Z}$. It turns out (see Xu and Yung [20]; Zwart [21]) that in the case when $\mathcal{A}$ generates a C_0-group (not only C_0-semigroup) satisfying (B1)–(B2), the condition (B3) is a consequence of weaker condition:

(B3') the span over the (generalized) eigenvectors of $\mathcal{A}$ is dense in $\mathcal{H}$.

The main goal of our paper is to extend polynomial stability analysis to the mentioned above class of C_0-semigroups. We obtain a spectral criterion for polynomial stability of not necessarily bounded semigroups generated by the operators satisfying (B1)–(B3). In particular, we describe the asymptotic behavior of the semigroups restricted to some dense, non-closed subsets in terms of location of the spectrum. Thus we obtain

Theorem 1.1 *Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ generate C_0-semigroup $T(t)$ on $\mathcal{H}$ and satisfy assumptions (B1)–(B3). If*

$$(A') \quad \mathrm{Re}\,\lambda - \omega_0 \leq -\gamma|\mathrm{Im}\lambda|^{-\alpha} \text{ for all } \lambda \in \sigma(\mathcal{A})$$

for some real, positive constants α, γ, then

$$\begin{aligned} &(a)\ \|R(\mathcal{A}, is + \omega_0)\| = O(|s|^{\alpha N}), \quad s \in \mathbb{R}, \quad s \to \infty, \\ &(b)\ \|T(t)(\mathcal{A} - \omega_0 I)^{-n}\| = O(e^{\omega_0 t} t^{N-1-\frac{n}{\alpha}}), \quad t \to +\infty. \end{aligned}$$

Basing on this Theorem and results from [2] we obtain the following:

Theorem 1.2 *Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ generate C_0-semigroup $T(t)$ on $\mathcal{H}$ and satisfies assumptions (B1)–(B3) and let $\rho(\mathcal{A}) \subset \mathbb{C}^-$. Then semigroup $T(t)$ is polynomially stable if and only if condition (A) holds for some positive constants γ, α and small values of* $|\mathrm{Re}\,\lambda|$.

We also use Theorem 1.1 to describe the asymptotic behavior of solutions of neutral type equations (Theorem 3.1). This theorem complements our previous results [16] concerning the behavior of the norm of semigroups corresponding to equations (3).

The work is organized as follows. First we give the proof of Theorem 1.1 preceded by several technical results. Next section is devoted to the analysis of stability of neutral type equations (3) and regular feedback stabilizability of these equations [14]. In the appendix we give two simple statements about complex matrices, which are used in our work.

2 Proof of the Main Results

In the beginning we give some technical results which will be used in the proof of Theorem 1.1.

Lemma 2.1 *For any sequence* $\{\lambda_1, \dots, \lambda_n, \hat{\lambda}_1, \dots, \hat{\lambda}_n\}$ *of* $2n$ *pairwise different complex numbers and any complex number* $\tilde{\lambda}$ *the system of linear equations*

$$\sum_{k=1}^{n} \frac{\alpha_k}{\lambda_i - \hat{\lambda}_k} = \lambda_i - \tilde{\lambda}, \quad i = 1, 2, \dots, n, \tag{7}$$

with n *unknowns* $\alpha_1, \dots, \alpha_n$ *has a unique solution given by the formula*

$$\alpha_k = \left(\hat{\lambda}_k - \tilde{\lambda} + \sum_{p=1}^{n}(\lambda_p - \hat{\lambda}_p)\right)\left(\lambda_k - \hat{\lambda}_k\right) \prod_{p=1;\, p\neq k}^{n} \frac{\lambda_p - \hat{\lambda}_k}{\hat{\lambda}_p - \hat{\lambda}_k}, \quad k = 1, 2, \dots, n. \tag{8}$$

Proof We solve the system by Cramer's rule. It is easy to compute the main determinant D and the determinant D_j that is determinant D, with j-th column replaced with the right hand side of system (7). Namely we have

$$D = \frac{\prod_{i>j}(\lambda_i - \lambda_j)(\hat{\lambda}_j - \hat{\lambda}_i)}{\prod_{i,j=1}^{n}(\lambda_i - \hat{\lambda}_j)},$$

$$D_k = \frac{\prod_{i>j}(\lambda_i - \lambda_j)\prod_{i>j;i,j\neq k}(\hat{\lambda}_j - \hat{\lambda}_i)}{\prod_{i,j=1;\,j\neq k}^{n}(\lambda_i - \hat{\lambda}_j)} \left(\hat{\lambda}_k - \tilde{\lambda} + \sum_{p=1}^{n}(\lambda_p - \hat{\lambda}_p)\right).$$

Taking $\alpha_k = \frac{D_k}{D}$, $k = 1, \dots, n$ we arrive at (8). □

Lemma 2.2 *Let* $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$*, be a generator of* C_0*-semigroup* $e^{\mathcal{A}t}$ *satisfying assumptions (B1)–(B3). In addition to this, we assume*

(B1') *the spectral families* σ_k *are vertically separated i.e. there exists a positive constant* d_v *such that*

$$\inf\{|\mathrm{Im}\lambda - \mathrm{Im}\mu| : \lambda \in \sigma_i, \mu \in \sigma_j, i \neq j\} = d_v > 0.$$

Then there exists constant $M > 0$ *independent of* k *such that* $\|e^{\mathcal{A}_k t}\| \leq M e^{\omega_k t}(t^{N-1} + 1)$*, where* $\omega_k = \max\{\mathrm{Re}\,\lambda : \lambda \in \sigma_k\}$.

Proof We define the operator $\mathcal{B} : D(\mathcal{A}) \to \mathcal{H}$ in each subspace V_k separately, by the following formula:

$$\mathcal{B}|_{V_k} x_k := \mathcal{B}_k x_k := \mathcal{A}_k x_k + (\omega_0 - \omega_k)x_k, \quad k \in \mathbb{Z}, \quad x = \sum_{k\in\mathbb{Z}} x_k,$$

where $\omega_0 = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\mathcal{A})\}$. Then for each $k \in \mathbb{Z}$ we have $\max\{\operatorname{Re}\lambda : \lambda \in \sigma(\mathcal{B}_k)\} \leq \omega_0$. Operator $\mathcal{B}$ generates C_0-semigroup and it satisfies the assumptions (B1)–(B3). Hence (see [9, Theorem 1d])

$$\|e^{\mathcal{B}t}\| \leq Me^{\omega_0 t}(t^{N-1}+1), \quad t \geq 0,$$

and in the subspaces V_k we get

$$\|e^{\mathcal{A}_k t}\| \cdot e^{(\omega_0-\omega_k)t} = \|e^{\mathcal{B}_k t}\| \leq Me^{\omega_0 t}(t^{N-1}+1), \quad t \geq 0,$$

which implies the assertion. □

Theorem 2.1 *Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$, satisfy assumptions (B1)–(B3) and generates C_0-semigroup on $\mathcal{H}$. If there exists a constant $L > 0$ with*

$$\omega_0 - \operatorname{Re}\lambda \leq L < +\infty, \quad \lambda \in \sigma(\mathcal{A}), \tag{9}$$

then there exists a constant M independent of k such that

$$\|\mathcal{A}_k - \tilde{\lambda}_k I\| \leq M, \quad k \in \mathbb{Z},$$

where $\mathcal{A}_k$ denotes the restriction of $\mathcal{A}$ to the corresponding basis subspace $V_k := \mathcal{P}_k\mathcal{H}$, $k \in \mathbb{Z}$ and $\tilde{\lambda}_k \in \sigma_k$ is an eigenvalue from σ_k with maximal real part.

Proof Without loss of generality we assume that each $\mathcal{A}_k$ has $n_k \leq N$ different eigenvalues and no rootvectors. Indeed, if this assumption is not satisfied then for any $\varepsilon_1 > 0$ we can find the operator $\mathcal{A}_\varepsilon$ close to $\mathcal{A}$ e.i. satisfying condition $\|\mathcal{A}_\varepsilon - \mathcal{A}\| \leq \varepsilon_1$ and $V_k' = V_k$, $k \in \mathbb{Z}$, and $\mathcal{A}_\varepsilon$ has only simple, different eigenvalues. If assertion is true for any operator $\mathcal{A}_\varepsilon$, $\varepsilon > 0$ then it is also true for $\mathcal{A}$.

Existence of Riesz basis of subspaces allows us to consider operator $\mathcal{A}$ and its resolvent in each subspace separately. We remind our notation $\mathcal{A}_k := \mathcal{A}|_{V_k}$ and $\mathcal{R}_k(\mathcal{A},\lambda) := \mathcal{R}(\mathcal{A},\lambda)|_{V_k}$. In each subspace V_k we choose an orthonormal system $\{e_i^{(k)}\}_{i=1}^{n_k}$. Note that the system $\{e_i^{(k)} : i = 1,\ldots,n_k; k \in \mathbb{Z}\}$ constitute a Riesz basis in $\mathcal{H}$. We define a family of matrices $P_k \in M_{n_k}(\mathbb{C}), k \in \mathbb{Z}$, as $P_k = [a_{ij}^{(k)}]$, where $a_{ij}^{(k)}$ are the coefficients of normalized eigenvector $v_i^{(k)}$ in the basis $\{e_j^{(k)} : j = 1,\ldots,n_k\}$. Let us denote the matrices of the operators $\mathcal{A}_k$ and $\mathcal{R}_k(\mathcal{A},\lambda)$ in the basis $\{e_j^{(k)} : j = 1,\ldots,n_k\}$ by A_k and $R_k(A,\lambda)$, respectively. Thus we have

$$A_k - \tilde{\lambda}_k I = P_k \Delta_k(\tilde{\lambda}_k) P_k^{-1}, \tag{10}$$

where $\Delta_k(\tilde{\lambda}_k)$ is a diagonal matrix with entries $\{\lambda_1^{(k)} - \tilde{\lambda}_k, \lambda_2^{(k)} - \tilde{\lambda}_k, \ldots, \lambda_{n_k}^{(k)} - \tilde{\lambda}_k\}$ and

$$R_k(A,\lambda) = P_k \Delta_k^{-1}(\lambda) P_k^{-1}, \tag{11}$$

where $\Delta_k^{-1}(\lambda)$ is a diagonal matrix with entries $\{(\lambda_1^{(k)}-\lambda)^{-1}, (\lambda_2^{(k)}-\lambda)^{-1}, \ldots, (\lambda_{n_k}^{(k)} - \lambda)^{-1}\}$.

$\mathcal{A}$ generates a C_0-semigroup $T(t)$, thus there exist constants M_1, ω_0 such that

$$\|T(t)\| \leq M_1 e^{(\omega_0+1)t}, t \geq 0 \quad \text{and} \quad \|\mathcal{R}(\mathcal{A}, \lambda)\| \leq \frac{M_1}{\operatorname{Re}\lambda - (\omega_0+1)}, \quad \operatorname{Re}\lambda > 1. \tag{12}$$

Using Riesz basis property we can conclude the same for $\|R(A_k, \lambda)\|$:

$$\|R(A_k, \lambda)\| \leq \frac{M_2}{\operatorname{Re}\lambda - (\omega_0+1)}, \quad \operatorname{Re}\lambda > \omega_0 + 1, \quad k \in \mathbb{Z}. \tag{13}$$

To estimate $\|A_k - \tilde{\lambda}_k I\|$, we decompose $\Delta_k(\tilde{\lambda}_k)$ as follows:

$$\Delta_k(\tilde{\lambda}_k) = \sum_{j=1}^{n_k} \alpha_j^{(k)} \Delta_k^{-1}(\hat{\lambda}_j^{(k)}), \tag{14}$$

where $\{\hat{\lambda}_j^{(k)}\}_{j=1}^{n_k}$ is any pairwise different complex sequence such that $\hat{\lambda}_j^{(k)} \neq \lambda_i^{(k)}$ for $i, j = 1, \ldots, n_k$. We find coefficients $\alpha_j^{(k)}$, $j = 1, 2, \ldots, n_k$; $k \in \mathbb{Z}$ by Lemma 2.1 namely, we have

$$\alpha_j^{(k)} = \left(\hat{\lambda}_j^{(k)} - \tilde{\lambda}_k + \sum_{p=1}^{n_k}(\lambda_p^{(k)} - \hat{\lambda}_p^{(k)})\right)\left(\lambda_j^{(k)} - \hat{\lambda}_j^{(k)}\right) \prod_{p=1;\, p\neq j}^{n_k} \frac{\lambda_p^{(k)} - \hat{\lambda}_j^{(k)}}{\hat{\lambda}_p^{(k)} - \hat{\lambda}_j^{(k)}}. \tag{15}$$

We choose $\hat{\lambda}_p^{(k)} := i\operatorname{Im}\tilde{\lambda}_k + \omega_0 + p + 1$ for $p = 1, 2, \ldots n_k$, $k \in \mathbb{Z}$ and observe that $|\hat{\lambda}_p^{(k)} - \hat{\lambda}_j^{(k)}| < n_k \leq N$. Without loss of generality we assume that the radius of each set σ_k is uniformly bounded by a constant $r > 0$. Hence $|\lambda_p^{(k)} - \hat{\lambda}_j^{(k)}| \leq N + L + 2r$. Taking the above into account we rewrite equality (15) to the form

$$|\alpha_j^{(k)}| \leq (N+1)(2r + L + N)^{N+1} =: M(N, r, L). \tag{16}$$

Now we estimate the norm of $A_k - \tilde{\lambda}_k I$. From (10) and (14) we get

$$\|A_k - \tilde{\lambda}_k I\| \leq \sum_{j=1}^{n_k} |\alpha_j^{(k)}| \|P_k \Delta_k^{-1}(\hat{\lambda}_j^{(k)}) P_k^{-1}\|, \quad k \in \mathbb{Z}.$$

Next we use (11) and inequality (13) for $\lambda = \hat{\lambda}_j^{(k)}$ in the above to imply

$$\|A_k - \tilde{\lambda}_k I\| \leq \sum_{j=1}^{n_k} |\alpha_j^{(k)}| \frac{M_2}{j}, \quad k \in \mathbb{Z}.$$

Estimation (16) and $n_k \leq N$ gives

$$\|A_k - \tilde{\lambda}_k I\| \leq M_2 N M(N, r, L), \quad k \in \mathbb{Z},$$

that completes the proof. □

Lemma 2.3 *Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ satisfy assumptions (B1)–(B3) and generate a C_0-semigroup on $\mathcal{H}$. Then for a sufficiently small $\varepsilon > 0$ and $I_\varepsilon := \{k \in \mathbb{Z} : \sigma_k \cap (\operatorname{Re}\lambda > \omega_0 - \varepsilon) \neq \emptyset\}$, there exists a constant M independent of k such that*

$$\|e^{A_k t}\| \leq M e^{\omega_k t}(t^{N-1} + 1), \quad t > 0, \quad k \in I_\varepsilon,$$

and

$$\|R(\lambda, A_k)\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega_k)^N} + \frac{M}{(\operatorname{Re}\lambda - \omega_k)}, \quad \operatorname{Re}\lambda > \omega_k, \quad k \in I_\varepsilon.$$

where $\omega_0 = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\mathcal{A})\}$, $\omega_k = \max\{\operatorname{Re}\lambda : \lambda \in \sigma_k\}$ and A_k is a matrix of the operator $\mathcal{A}$ projected to the subspace V_k.

Proof We choose ε small enough for the families $\sigma_k' := \sigma_k - \omega_k, k \in I_\varepsilon$, satisfy assumptions (B1). Then the subspaces V_k, $k \in I_\varepsilon$ constitute a Riesz basis of subspaces in the closure of a corresponding linear span, which can be extended to the Riesz basis (in this span) by choosing an orthonormal basis in each subspace. In this basis we consider the matrices A_k of the operators $\mathcal{A}_k := \mathcal{A}|_{V_k}$. We define a new operator $\mathcal{B}$ in each subspace V_k, namely we take $\mathcal{B}_k := \mathcal{B}|_{V_k} := \mathcal{A}_k + (\omega_0 - \omega_k)I$. It is easy to see that the operator $\mathcal{B}$ still generates a C_0-semigroup and $\omega_0(\mathcal{B}) = \omega_0(\mathcal{A}) = \omega_0$. Theorem 1 in [9] implies that

$$\|e^{\mathcal{B}t}\| \leq M_2 e^{\omega_0 t}(t^{N-1} + 1), \quad t > 0,$$

for some constant M_2. Hence for the operator $e^{\mathcal{B}_k t}$ and its matrix $e^{B_k t}$ we have a similar estimation

$$\|e^{B_k t}\| \leq M_1 e^{\omega_0 t}(t^{N-1} + 1), \quad t > 0, k \in I_\varepsilon,$$

with a new constant M_1. From the definition of $\mathcal{B}_k$ and the last inequality we conclude that

$$\|e^{A_k t}\| \leq M_1 e^{\omega_k t}(t^{N-1} + 1), \quad t > 0, k \in I_\varepsilon,$$

which proves the first assertion. Now we estimate the norm of the resolvent operator $R(A_k, \lambda)$ using a Laplace transform of $e^{A_k t}$, namely

$$R(A_k, \lambda)x = \int_0^{+\infty} e^{-\lambda t} e^{A_k t} x dt, \quad \operatorname{Re}\lambda > \omega_k.$$

For the norms we get

$$\|R(A_k, \lambda)x\| \leq \int_0^{+\infty} |e^{-\lambda t}| \cdot \|e^{A_k t} x\| dt \leq \int_0^{+\infty} M_1 e^{-(\operatorname{Re}\lambda - \omega_k)t}(t^{N-1} + 1)\|x\| dt.$$

This finally gives

$$\|R(A_k,\lambda)x\| \leq \frac{M_2\|x\|}{(\operatorname{Re}\lambda-\omega_k)^N}+\frac{M_2\|x\|}{\operatorname{Re}\lambda-\omega_k}, \quad \operatorname{Re}\lambda>\omega_k,$$

where M_2 is a constant independent of k. □

Now we are ready to prove the main theorem.

Proof Without loss of generality we assume that $\omega_0=0$. To prove (a) we generate a basis in $\mathcal{H}$, by taking the Riesz basis from subspaces and choosing an orthonormal basis in each of the subspace. Obviously, such a basis is a Riesz basis and we can consider matrices $R(A_k,\lambda)$ of the resolvent operators in this basis instead of the operators $R(\mathcal{A}_k,\lambda)$ themselves. We prove that for some constant $M_1>0$

$$\sup_k \|R(A_k,is)\| \leq M_1|s|^{N\alpha}, \quad s\to\infty. \tag{17}$$

First, we split the set of all k's into three subsets $I_0(\varepsilon):=\mathbb{Z}\setminus I_\varepsilon$, $I_1(s,C,\varepsilon):=\{k\in I_\varepsilon : |\tilde{\lambda}_k-is|<C\}$ and $I_2(s,C,\varepsilon):=\{k\in I_\varepsilon : |\tilde{\lambda}_k-is|\geq C\}$, where $I_\varepsilon:=\{k\in\mathbb{Z} : \sigma_k\cap(\operatorname{Re}\lambda>\omega_0-\varepsilon)\neq\emptyset\}$. Second, we choose ε small enough to use Lemma 2.3. For $k\in I_0$ the assertion is obvious because the restricted semigroup is bounded by the exponent $M_\varepsilon e^{-\frac{1}{2}\varepsilon t}$, and we need to prove it only for $k\in I_\varepsilon$. Theorem 2.1 implies $\frac{1}{M}(A_k-isI)$ is close to $\frac{1}{M}(\tilde{\lambda}_k-is)I$, i.e. $\left\|\frac{1}{M}(A_k-isI)-\frac{1}{M}(\tilde{\lambda}_k-is)I\right\|\leq 1, k\in\mathbb{Z}$. We estimate $\|R(A_k,is)\|$ for $k\in I_2(s,C,\varepsilon)$ using Statement 3.5 (see Appendix) for family of matrices $\frac{1}{M}(A_k-isI)$. We fix a constant C (independently of s) large enough to make sure that $\frac{1}{M}|\tilde{\lambda}_k-is|$ satisfy assumptions of Statement 3.5 for all $k\in I_2(C,s,\varepsilon)$. Hence we have

$$\|R(A_k,is)\| \leq \frac{C_1}{|\tilde{\lambda}_k-is|}\leq\frac{C_1}{C}, \quad k\in I_2, \tag{18}$$

where constant C_1 is independent of k.

For $k\in I_1(C,s,\varepsilon)$, we use Lemma 2.3,

$$\|R(A_k,\lambda)\| \leq \frac{M}{|\operatorname{Re}\lambda-\omega_k|^N}+\frac{M}{|\operatorname{Re}\lambda-\omega_k|}, \quad \operatorname{Re}\lambda>\omega_k,$$

where constant M is independent of k. Taking $\lambda=is$ and using the assumption (A) we get

$$\|R(A_k,is)\| \leq \frac{M}{|\omega_k|^N}\leq |Im\tilde{\lambda}_k|^{N\alpha}\leq M|s+C|^{N\alpha}\leq M_2|s|^{N\alpha}, \quad k\in I_1. \tag{19}$$

Combining (18) and (19) we get (17) which proves the assertion (a).

To prove (b) it suffices to show

$$\sup_{k\in\mathbb{Z}} \|T_k(t)\mathcal{A}_k^{-n}\| \le M t^{N-1-\frac{n}{\alpha}}, \quad t > 1. \tag{20}$$

For $\varepsilon > 0$ small enough and any $C > 0$ it is easy to see that

$$\|T_k(t)\mathcal{A}_k^{-n}\| \le M_\varepsilon e^{-\frac{1}{2}\varepsilon t} \le M'_\varepsilon t^{N-1-\frac{n}{\alpha}}, \quad t > 1, k \in I_0 \cup I_1(0, C, \varepsilon). \tag{21}$$

For $k \in I_2(0, C, \varepsilon)$ we use Lemma 2.3 and obtain

$$\|T_k(t)\mathcal{A}_k^{-n}\| \le M_1 e^{\omega_k t} t^{N-1} \|A_k^{-1}\|^n, \quad t > 1, k \in I_2(0, C, \varepsilon).$$

For $|\tilde{\lambda}_k|$ large enough we use Statement 3.5 to estimate $\|A_k^{-1}\|$ in the same way as in the proof of assertion (a) i.e. $\|A_k^{-1}\| \le \frac{M}{|\tilde{\lambda}_k|}$. Hence we obtain

$$\|T_k(t)\mathcal{A}_k^{-n}\| \le M_2 e^{\omega_k t} t^{N-1} |\tilde{\lambda}_k|^{-n}, \quad t > 1, k \in I_2(0, C, \varepsilon).$$

From assumption (A) we have $|\omega_k| = |\mathrm{Re}\,\tilde{\lambda}_k| \ge \gamma |\mathrm{Im}\tilde{\lambda}_k|^{-\alpha} \ge \gamma |\tilde{\lambda}_k|^{-\alpha}$, so

$$\|T_k(t)\mathcal{A}_k^{-n}\| \le M_2 e^{\omega_k t} t^{N-1} |\omega_k|^{\frac{n}{\alpha}} \le M_2 e^{\omega_k t} t^{N-1} |\omega_k t|^{\frac{n}{\alpha}} t^{-\frac{n}{\alpha}}, \quad t > 1.$$

For each $\beta > 0$ the function $e^{-x}x^\beta$, $x \ge 0$, is bounded, hence

$$\|T_k(t)\mathcal{A}_k^{-n}\| \le M_3 t^{N-1} t^{-\frac{n}{\alpha}}, \quad t > 1, k \in I_2, \tag{22}$$

where M_3 is a new constant depending on n, α. Now (21) and (22) imply (20) and assertion (b) is proven. □

Proof Theorem 1.1 shows that condition (A) is sufficient for polynomial stability. Necessity. Let $T(t)$ be polynomially stable. We choose one eigenvalue from each family σ_k (say λ_k) and corresponding eigenvectors ϕ_k. Consider subspace $S = \overline{\mathrm{span}}\,\{\phi_k : k \in \mathbb{Z}\}$. It is easy to see that subspace S is T-invariant and the semigroup $T(t)$ is bounded on S. Applying the results of Sect. 3 from [2] for semigroup $T(t)$ restricted to S we see that family $\{\lambda_k\}_{k\in\mathbb{Z}}$ satisfies condition (A) with some positive constants γ, α.

Since eigenvalues λ_k was chosen arbitrarily, then whole spectrum satisfies condition (A) with some positive constants γ, α. □

3 Stability and Stabilizability of Neutral Type Equations

Following [13], we consider the delay systems of neutral type of the form (3), which can be represented in the operator form (4), with generator $\mathcal{A}$ given by (5)–(6). Our goal is to investigate the asymptotic behavior of the solutions of above equation, in particular its stability. The stability is closely related to the location of the spectrum of the operator $\mathcal{A}$ thus we recall some important properties of $\mathcal{A}$ (for more details see [13]).

We denote the eigenvalues of the matrix A_{-1} by $\mu_m, m = 1, \dots, \ell$ ($|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_\ell|$), and their multiplicities by p_m ($\sum p_m = n$). Without loss of generality we assume that if $|\mu_1| = |\mu_2| = \cdots = |\mu_{\ell_0}|$ then $p_1 \geq p_2 \geq \cdots \geq p_{\ell_0}$. The spectrum of $\mathcal{A}$ cannot be determined explicitly in general, but it is close to the spectrum of the operator $\tilde{\mathcal{A}}$, which appears when we put $A_2 = A_3 = 0$ in (5). The eigenvalues of $\tilde{\mathcal{A}}$ are complex logarithms of μ_m and zero, i.e.

$$\sigma(\tilde{\mathcal{A}}) = \{\tilde{\lambda}_m^{(k)} = \ln|\mu_m| + i(\arg\mu_m + 2k\pi), \mu_m \in \sigma(A_{-1}), m = 1, \dots, \ell; k \in \mathbb{Z}\} \cup \{0\}.$$

We denote $\max\{\operatorname{Re}\lambda : \lambda \in \sigma(\tilde{\mathcal{A}})\}$ by $\tilde{\omega}$ and $\sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\mathcal{A})\}$ by ω.

The spectrum of $\mathcal{A}$ consist of eigenvalues only. Almost all of them lie close to $\tilde{\lambda}_m^{(k)}$. More precisely, for k large enough they are contained in the discs $L_m^{(k)}$ centered at $\tilde{\lambda}_m^{(k)}$ of radii $r_k \to 0$ (see [14, Theorem 4]). The sum of multiplicities of eigenvalues of $\mathcal{A}$ lying in each disc centered at $\tilde{\lambda}_m^{(k)}$ equals the multiplicity of $\tilde{\lambda}_m^{(k)}$ and μ_m, that is p_m. We denote eigenvalues of the operator $\mathcal{A}$ by $\lambda_{m,i}^{(k)}$, $k \in \mathbb{Z}$; $m = 1, \dots, \ell$, and we have $\{\lambda_{m,i}^{(k)}\}_{i=1}^{p_m} \subset L_m^{(k)}$, $|k| > N$; $m = 1, \dots, \ell$. If there exist eigenvalues of $\mathcal{A}$ with real part $\omega = \sup_{\lambda\in\sigma(\mathcal{A})} \operatorname{Re}\lambda$ we denote their maximal multiplicity by p_0 and we set $p_0 = 0$ if there are no such eigenvalues. Let us denote $\mathcal{A}$-invariant subspaces $V_m^{(k)} = P_m^{(k)}\mathcal{M}_2$, where $P_m^{(k)}x = \frac{1}{2\pi i}\int_{L_m^{(k)}} R(\mathcal{A}, \lambda)x d\lambda$ are Riesz projections, $m = 1, \dots, \ell, k \in \mathbb{Z}$. The sequence of p_m-dimensional subspaces $V_m^{(k)}$, $m = 1, \dots, \ell$, $|k| \geq N$, and some $2(N+1)n$-dimensional subspace W_N constitute an $\mathcal{A}$-invariant Riesz basis of space $\mathcal{M}_2$.

Taking above into the account, we get $\omega \geq \tilde{\omega}$ and there is a few possibilities for location of the spectrum $\sigma(\mathcal{A})$:

(a) $\omega > \tilde{\omega}$, which implies that $p_0 > 0$;

(b) $\omega = \tilde{\omega}$ and $p_0 = 0$.

(c) $\omega = \tilde{\omega}$ and $0 < p_0 < q$, where $q \geq 1$ is maximal size of Jordan block of matrix A_{-1} corresponding to the eigenvalue μ_1;

(d) $\omega = \tilde{\omega}$ and $q \leq p_0 < p_1$, where p_1 is the multiplicity of μ_1;

(e) $\omega = \tilde{\omega}$ and $p_0 \geq p_1$.

In the cases (a) and (e) an asymptotic behavior of the corresponding semigroup is determined by the eigenvalue with maximal real part (equals ω) and multiplicity p_0 i.e.

$$\|e^{\mathcal{A}t}\| \leq Me^{\omega t}t^{p_0-1}, \quad t > 1.$$

In the cases (b)–(d) we have the following estimation for the norm of semigroup (see [16] for more details):

$$me^{\tilde{\omega}t}t^{q-1} \leq \|e^{\mathcal{A}t}\| \leq Me^{\tilde{\omega}t}t^{p_1-1}, \quad t > 1. \tag{23}$$

Moreover in the cases (b) and (c) the semigroup does not have any maximal asymptotics (even when $q = p_1$) i.e.

$$\lim_{t\to+\infty} \frac{\|e^{\mathcal{A}t}x\|}{\|e^{\mathcal{A}t}\|} = 0, \quad x \in \mathcal{M}_2, \tag{24}$$

in the case (d), an existence of the maximal asymptotics is independent of our assumptions.

Now we discuss the property of asymptotic stability in the above cases. The necessary condition of stability is that $\omega \leq 0$ and if $\omega < 0$ then it is even exponential stability, thus only the case $\omega = 0$ is interesting. In this case we see that stability cannot occur in (a), (c), (d), (e) because we can point out an initial state for which the solution does not decrease or, at least, such an initial state exists (by Banach–Steinhaus Theorem). We focus on the case (b), where $\omega = 0$ and $p_0 = 0$ and discuss the stability. If $q = p_1 = 1$ then (23) and lack of maximal asymptotics (24) implies stability. For $q > 1$ the lack of stability is a consequence of inequality (23) and it is caused by the families of eigenvalues approaching imaginary axis from the left-hand side (not by the single eigenvalue). If it is possible to describe the rate of this approaching using the following inequality

$$\operatorname{Re}\lambda \leq -\frac{C}{|\operatorname{Im}\lambda|^{\alpha}}, \quad \lambda \in \sigma(\mathcal{A}), \tag{25}$$

where C, α are some real positive constants, then we are able to find a subset of initial states for which the system is stable. Using Theorem 1.1 we obtain sufficient condition for the stability of the system on some non-closed subset, namely we have

Theorem 3.1 *Let us consider system* (4). *If* $\operatorname{Re}\lambda < 0$ *for all* $\lambda \in \sigma(\mathcal{A})$ *and* $\operatorname{Re}\lambda \leq -\frac{C}{|\operatorname{Im}\lambda|^{\alpha}}$ *for all but finitely many* $\lambda \in \sigma(\mathcal{A})$, *where* C, α *are some real positive constants, then for any* $n \in \mathbb{N}$ *there exists* $M > 0$ *with*

$$|e^{\mathcal{A}t}\mathcal{A}^{-n}x\| \leq Mt^{p-1-\frac{n}{\alpha}}\|x\|, \quad t > 1, \quad x \in \mathcal{M}_2.$$

Proof of Theorem 1.2 The operator $\mathcal{A}$ satisfies the assumptions (B1)–(B3) and (A), so the proof of theorem follows directly from Theorem 1.1. □

Corollary 3.1 *For the system* (4) *satisfying assumption of Theorem 3.1 and any constant* $\beta > 0$ *there exists* n_0 *large enough and a constant* $M > 0$ *such that*

$$\|e^{\mathcal{A}t}x\| \leq \frac{M\|x\|_{D(\mathcal{A}^{n_0})}}{t^{\beta}}, \quad t > 1, \quad x \in D(\mathcal{A}^{n_0}),$$

where $\|\cdot\|_{D(\mathcal{A}^{n_0})}$ *denotes the norm* $\|x\|_{D(\mathcal{A}^{n_0})} = \|\mathcal{A}^{n_0}x\| + \|x\|$.

Now, following [14] we consider regular feedback stabilizability of a system

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + \int_{-1}^{0} A_2(\theta)\dot{z}(t+\theta)d\theta + \int_{-1}^{0} A_3(\theta)z(t+\theta)d\theta + Bu, \quad (26)$$

where A_{-1} is an $n \times n$ invertible complex matrix, A_2 and A_3 are $n \times n$ matrices of functions from $L_2(-1, 0)$, B is an $n \times p$ complex matrix, $z(t + \cdot) \in H^1(-1, 0; \mathbb{C}^n)$. It was shown in [7,11] that for any $u \in L_2$ the system (26) has the unique solution $z(t+\cdot) \in H^1(-1, 0; \mathbb{C}^n)$. We say that the system (26) is asymptotically stabilizable if there exists a linear feedback control $u(t) = F(z_t(\cdot)) = F(z(t+\cdot)$ such that the system (26) becomes asymptotically stable. If in addition to this the asymptotic stabilizability is achieved by a feedback F which is bounded (as an operator acting on space H^1), then we call it regular asymptotic stabilizability. In our case any regular feedback is of the form (see [14] for more details)

$$u(t) = F(z_t(\cdot)) = F(z(t+\cdot)) = \int_{-1}^{0} F_2(\theta)\dot{z}(t+\theta)d\theta + \int_{-1}^{0} F_3(\theta)z(t+\theta)d\theta, \quad (27)$$

where $F_2(\cdot), F_3(\cdot) \in L_2([-1, 0], \mathbb{C}^{n\times p})$. The Eq. (26) can be rewritten in the operator form similar to the Eq. (3),

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \qquad x \in \mathcal{M}_2, \quad (28)$$

where operator $\mathcal{A}$ is given by (5)–(6), and $\mathcal{B}u = \binom{Bu}{0}$. Taking (27) into account we can rewrite the Eq. (28) in the form

$$\dot{x} = (\mathcal{A} + \mathcal{B}\mathcal{F})x, \qquad x \in \mathcal{M}_2. \quad (29)$$

We notice that $\mathcal{A} + \mathcal{B}\mathcal{F}$ is similar to the operator $\mathcal{A}$. The operator $\mathcal{B}\mathcal{F}$ affects only the matrices A_2, A_3 and the operator $\mathcal{A} + \mathcal{B}\mathcal{F}$ is of the same form as $\mathcal{A}$ with only A_2 and A_3 exchanged. In particular, the operator $\mathcal{A} + \mathcal{B}\mathcal{F}$ generates a C_0-group and its domain stays unchanged (because the operator $\mathcal{B}\mathcal{F}$ does not affect the matrix A_{-1}). In the case when eigenvalues of the matrix A_{-1} with maximal modulus, say $\mu_m, m = 1, \ldots, \ell_0$, are different and simple, the corresponding eigenvalues of $\mathcal{A}$, say $\{\lambda_k\}_{k\in\mathbb{Z}}$ (and $\mathcal{A} + \mathcal{B}\mathcal{F}$), are also simple and are in some disjoint circles of summable with square radii r_k. It was proven (see [14, Theorem 8] and [18,19]) that in such a case for any choice of complex sequence $\{\hat{\lambda}_k\}_{k\in\mathbb{Z}}$ in the same circles, there exists feedback $\mathcal{F}$ of the form (27) such that the numbers $\hat{\lambda}_k$ will be eigenvalues of $\mathcal{A} + \mathcal{B}\mathcal{F}$. In other words, the eigenvalues of $\mathcal{A}$ can be moved by the feedback to any point of corresponding circles. In particular, if centers of circles are on the imaginary axis then eigenvalues of $\mathcal{A}$ can be moved to the left open half-plane and the C_0-group generated by $\mathcal{A} + \mathcal{B}\mathcal{F}$ will be stable (i.e. $e^{(\mathcal{A}+\mathcal{B}\mathcal{F})t} \leq M$). The following statement describes above situation

Statement 3.2 *Let us consider Eq. (28) with additional assumptions that*

(a) *spectrum of $\mathcal{A}$ consist of simple eigenvalues only, say $\sigma(\mathcal{A}) = \{\lambda_k : k \in \mathbb{Z}\}$,*

(b) *linear span of eigenvectors of $\mathcal{A}$ is dens in $\mathcal{H}$,*

(c) *eigenvalues λ_k lie in disjoint balls $B(x_k, r_k)$ centered at the following points of imaginary axis $x_k = i(kd + d_0)$, $0 \leq d_0 < d$ and radii r_k satisfies $\sum r_k^2 < \infty$,*

(d) *vector $b \in \mathcal{H}$ is not orthogonal to eigenvectors ϕ_k of the operator $\mathcal{A}^*$.*

Then for any $\alpha > \frac{1}{2}$ there exists regular feedback $\mathcal{F}$ of the form (27) such that the group $e^{(\mathcal{A}+b\mathcal{F})t}$ is:

(i) *stable, that is $\|e^{(\mathcal{A}+b\mathcal{F})t}\| < M$ for some constant $M > 0$,*

(ii) *polynomially stable, that is $\|e^{(\mathcal{A}+b\mathcal{F})t}\mathcal{A}^{-1}\| \leq M_\alpha t^{-\frac{1}{\alpha}}$ for some constant $M_\alpha > 0$.*

Proof of Theorem 1.2 By [14, Theorem 8,Lemma 13] there exist feedback $\mathcal{F}$, which shifts the eigenvalues λ_k to the points $\hat{\lambda} = -r_k' + (kd + d_0)i$, where $r_k' = \max\{r_k, Ck^{-\alpha}\}$. Then all $\hat{\lambda}_k$ are in the open left half-plane and assertion (i) follows from [9, Theorem 1]. To prove (ii) we check that eigenvalues $\hat{\lambda}_k$ satisfies condition (A') with constants $\omega_0 = 0, \alpha, \gamma = Cd^\alpha$ and use Theorem 1.1. □

For the case of non-single eigenvalues μ_m, $m = 1, \ldots, \ell_0$ of matrix A_{-1}, even if eigenvalues of $\mathcal{A}$ can be moved to the open left half-plane, then, in general, stability can not be obtained because the corresponding group can be unbounded (see [16]). Although if we assume that we are able to move eigenvalues in each circle using proper feedback (27) the same way like in the case of single eigenvalues, then using Theorem 1.1 we can obtain polynomial stability of a corresponding group. To illustrate this idea we focus on some special class of equations (26).

Let us denote identity matrix in $\mathbb{C}^n$ by I_n and Jordan block with eigenvalue 1 of size n by J_n, i.e. $J_n = \{a_{p,q}\} : a_{p,p} = a_{p,p+1} = 1$, $p = 1, \ldots, n$, and all other entries are 0. We consider the Eq. (26) with $A_{-1} = I_n$, $A_2 = \tilde{f}(\theta)J_n$, $A_3 = \tilde{g}(\theta)J_n$, where $\tilde{f}, \tilde{g} \in L_2(-1, 0)$ are fixed. We take $B = J_n$ and the control $u(t)$ of the form (27) i.e.

$$u(t) = \int_{-1}^{0} f_2(\theta)\dot{z}(t+\theta)d\theta + \int_{-1}^{0} f_3(\theta)z(t+\theta)d\theta,$$

where $f_2, f_3 \in L_2(-1, 0; \mathbb{C})$. Taking $f = \tilde{f} + f_2$, $g = \tilde{g} + f_3$ we rewrite the Eq. (26) in the form

$$\dot{z}(t) = I_n\dot{z}(t-1) + J_n\left(\int_{-1}^{0} f(\theta)\dot{z}_t(\theta)d\theta + \int_{-1}^{0} g(\theta)z_t(\theta)d\theta\right). \tag{30}$$

The corresponding characteristic function Δ is of the form

$$\Delta(\lambda) = \det\Big[I_n\left(\lambda e^{-\lambda} + \lambda\int_{-1}^{0} f(s)e^{\lambda s}\mathrm{d}s + \int_{-1}^{0} g(s)e^{\lambda s}\mathrm{d}s - \lambda\right) \\ + (J_n - I_n)\left(\lambda\int_{-1}^{0} f(s)e^{\lambda s}\mathrm{d}s + \int_{-1}^{0} g(s)e^{\lambda s}\mathrm{d}s - \lambda\right)\Big],$$

and it equals zero only if

$$\lambda e^{-\lambda} + \lambda\int_{-1}^{0} f(s)e^{\lambda s}\mathrm{d}s + \int_{-1}^{0} g(s)e^{\lambda s}\mathrm{d}s - \lambda = 0. \tag{31}$$

It is proven (see [13]) that roots of this equation are asymptotically close to roots of equation $\lambda(e^{\lambda} - 1) = 0$. More precisely the roots of equation (31) are in the circles centred in $\lambda_k = 2k\pi i$ and square summable radii r_k. For scalar version of equation (30) (i.e. $n = 1$) Theorem 8 in [14] implies that for any choice of complex sequence τ_k in the above circles there exist functions $f, g \in L_2$ such that the numbers τ_k will be roots of the equation (31). Moreover, the Eq. (31) does not depend on n, thus the same functions f, g move roots the same way in general case ($n > 1$). Now we choose $\tau_k = 2k\pi i - \frac{1}{k}$, what means that there exist functions f, g such that the numbers τ_k are eigenvalues of the corresponding operator $(\mathcal{A} + \mathcal{BF})$, whose eigenvalues are contained in the left open half-plane and satisfy (25) with $C = 2\pi$, $\alpha = 1$. Nevertheless the system (28) can not be stable because the corresponding group is not bounded i.e.

$$\|e^{(\mathcal{A}+\mathcal{BF})t}\| \geq Mt^{n-1}, \quad t > 1.$$

However, our paper provides tools to study polynomial stability of above unbounded group, in particular due to Theorem 3.1 we obtain that sufficiently regular solutions tend to zero polynomially. Namely we have the following

Statement 3.3 *We consider control system (28) with feedback control of the form (27). We fix $n \in \mathbb{N}_+$, $A_{-1} = I_n$, $A_2 = \tilde{f}(\theta)J_n$, $A_3 = \tilde{g}(\theta)J_n$, $F_2 = f_2(\theta)J_n$, $F_3 = f_3(\theta)J_n$, $B = J_n$, where $\tilde{f}, \tilde{g}$ are arbitrary functions from $L_2(-1, 0; \mathbb{C})$ and J_n is a Jordan block with eigenvalue 1 of size n. Then there exist functions $f_2, f_3 \in L_2(-1, 0; \mathbb{C})$ and constant $M > 0$ such that for any $k \in \mathbb{N}$*

$$\|e^{(\mathcal{A}+\mathcal{BF})t}(\mathcal{A} + \mathcal{BF})^{-(n+k-1)}\| \leq Mt^{-k}, \quad t > 1,$$

or equivalently

$$\|e^{(\mathcal{A}+\mathcal{BF})t}x\| \leq Mt^{-k}\|x\|_{D(\mathcal{A}^{n+k-1})}, \quad t > 1, x \in D\left(\mathcal{A}^{n+k-1}\right).$$

Acknowledgements The authors would like to thank the anonymous referee for the comments, which helped to improve the presentation of the paper and also would like to thank PROMEP (Mexico) via "Poyecto de Redes" for partial financial support of this research.

Appendix

Statement 3.4 *Let $A_\lambda = [a_{i,j}] \in M_n(\mathbb{C})$ be a Jordan block of eigenvalue λ, where $|\lambda| \geq 1$ and $B_\lambda = [b_{i,j}] \in M_n(\mathbb{C}) \in M_n(\mathbb{C})$ be such that $\|B_\lambda - A_\lambda\| \leq 1$, where $\|A\| = \sum |a_{i,j}|$, then there exists constant M such that*

$$|\det B_\lambda| \leq M|\lambda|^n,$$

and for $|\lambda| \geq 2M$ we have also

$$|\det B_\lambda| \geq \frac{1}{2}|\lambda|^n.$$

Proof of Theorem 1.2 Let us define $\varepsilon_{i,j} = b_{i,j} - a_{i,j}$, and renumerate them in any order $(\varepsilon_1, \ldots, \varepsilon_{n^2})$. It is easy to see that

$$|\det B_\lambda| = |\lambda^n + f_1(\varepsilon_1, \ldots, \varepsilon_{n^2})\lambda^{n-1} + f_2(\varepsilon_1, \ldots, \varepsilon_{n^2})\lambda^{n-2} + \cdots + f_n(\varepsilon_1, \ldots, \varepsilon_{n^2})\lambda^0|, \tag{32}$$

where $f_1, \ldots, f_n$ are polynomials. By the assumption that $\|B_\lambda - A_\lambda\| \leq 1$ we get $\sum |\varepsilon_{i,j}| \leq 1$ and therefore

$$M_0 := \sup\left\{|f_1(\varepsilon_1, \ldots, \varepsilon_{n^2})|, \ldots, |f_n(\varepsilon_1, \ldots, \varepsilon_{n^2})| : \sum |\varepsilon_{i,j}| \leq 1\right\}$$

is finite. Then by (32) we get

$$|\det B_\lambda| \leq |\lambda|^n + M_0 n|\lambda^n|, \tag{33}$$

where we used triangle inequality and $|\lambda|^i \leq |\lambda|^n, i = 0, 1, \ldots, n$. Taking $M = (n+1)M_0$ we get the first inequality. To prove the second one, we also use triangle inequality in (32) and obtain

$$|\det B_\lambda| \geq |\lambda|^n - M_0 n|\lambda|^{n-1}. \tag{34}$$

Hence for $|\lambda| \geq 2M$ we have

$$|\det B_\lambda| \geq \frac{1}{2}|\lambda|^n,$$

which ends the proof of the statement. □

Remark 1 With the same assumptions we can prove similarly the first inequality of Statement 3.4 for the cofactors of matrix B_λ. Namely, if we denote cofactors of matrix $B_\lambda \in M_n(\mathbb{C})$ by $B_{i,j}$, then $|B_{i,j}| \leq M|\lambda|^{n-1}$.

Statement 3.5 *Let $A_\lambda, B_\lambda \in M_n(\mathbb{C})$ be such that A_λ consist of Jordan blocks of eigenvalue λ, where $|\lambda|$ is sufficiently large and $\|B_\lambda - A_\lambda\| \leq 1$, where $\|A\| = \sum |a_{i,j}|$, then there exist constant C such that*

$$\|B_\lambda^{-1}\| \leq \frac{C}{|\lambda|}.$$

Proof of Theorem 1.2 Without loss of generality we assume that A_λ is a Jordan block. Using inversion formula to matrix B_λ we get

$$\left\| B_\lambda^{-1} \right\| = |\det B_\lambda|^{-1} \sum |B_{i,j}|,$$

where $B_{i,j}$ are cofactors of matrix B_λ. Using Statement 3.4 and Remark 1 to estimate $|\det B_\lambda|$ and $|B_{i,j}|$ we obtain

$$\|B_\lambda^{-1}\| \leq \frac{2}{|\lambda|^n} \cdot M|\lambda|^{n-1} = \frac{2M}{|\lambda|},$$

which ends the proof. □

References

1. Arendt, W., Batty, C.J.K.: Tauberian theorems and stability of one parameter semigroups. Trans. Am. Math. Soc. **306**, 837–852 (1988)
2. Bátkai, A., Engel, K.-J., Prüss, J., Schnaubelt, R.: Polynomial stability of operator semigroups. Math. Nachr. **279**, 1425–1440 (2006)
3. Batty, C.J.K.: Tauberian theorems for the Laplace-Stieltjes transform. Trans. Am. Math. Soc. **322**, 783–804 (1990)
4. Batty, C.J.K.: Asymptotic behaviour of semigroups of operators. In: Functional Analysis and Operator Theory (Warsaw, 1992), vol. 30, pp. 35–52. Banach Center Publications Polish Academy of Sciences, Warsaw (1994)
5. Borichev, A., Tomilov, Y.: Optimal polynomial decay of functions and operator semigroups. Math. Ann. **347**(2), 455–478 (2010)
6. Burns, J., Herdman, T., Stech, H.: Linear functional-differential equations as semigroups on product spaces. SIAM J. Math. Anal. **14**(1), 98–116 (1983)
7. Ito, K., Tarn, T.J.: A linear quadratic optimal control for neutral systems. Nonlinear Anal. Theory Methods Appl. **9**(7), 699–727 (1985)
8. Lyubich, Yu.I., Phong, V.Q.: Asymptotic stability of linear differential equation in Banach space. Stud. Math. **88**, 37–42 (1988)
9. Miloslavskii, A.I.: Stability of certain classes of evolution equations. Sib. Math. J. **26**, 118–132 (1985)
10. Phong, V.Q.: Theorems of Katznelson-Tzaffriri type for semigroups of operators. J. Funct. Anal. **103**, 74–84 (1992)
11. Rabah, R., Sklyar, G.M.: The analysis of exact controllability of neutral type systems by the moment problem approach. SIAM J. Control Optim. **46**(6), 2148–2181 (2007)
12. Rabah, R., Sklyar, G.M., Rezounenko, A.V.: Generalized Riesz basis property in the analysis of neutral type systems. C. R. Math. Acad. Sci. Paris **337**, 19–24 (2003)

13. Rabah, R., Sklyar, G.M., Rezounenko, A.V.: Stability analysis of neutral type systems in Hilbert space. J. Differ. Equ. **214**, 391–428 (2005)
14. Rabah, R., Sklyar, G.M., Rezounenko, A.V.: On strong regular stabilizability for linear neutral type systems. J. Differ. Equ. **245**, 569–593 (2008)
15. Sklyar, G.M.: On the decay of bounded semigroup on the domain of its generator. Vietnam J. Math. **43**, 207–213 (2015)
16. Sklyar, G.M., Polak, P.: Asymptotic growth of solutions of neutral type systems. Appl. Math. Optim. **67**(3), 453–477 (2013)
17. Sklyar, G.M., Shirman, V.Ya.: On the asymptotic stability of a linear differential equation in a Banach space. Teor. Funkc. Anal. Prilozh. (Kharkov) **37**, 127–132 (1982)
18. Sklyar, K.V., Rabah, R., Sklyar, G.M.: Eigenvalues and eigenvectors assignment for neutral type systems. C. R. Math. Acad. Sci. Paris **351**(3–4), 91–95 (2013)
19. Sklyar, K.V., Rabah, R., Sklyar, G.M.: Spectral assignment for neutral-type systems and moment problems. SIAM J. Control Optim. **53**(2), 845–873 (2015)
20. Xu, G.Q., Yung, S.P.: The expansion of a semigroup and a Riesz basis criterion. J. Differ. Equ. **210**(1), 1–24 (2005)
21. Zwart, H.: Riesz basis for strongly continuous groups. J. Differ. Equ. **249**, 2397–2408 (2010)